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The Significance of a Hidden Variable Proof and the Logical Interpretation of Quantum Mechanics[1]

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This paper examines the logical interpretation of quantum mechanics. Since this interpretation is based on a proof by Kochen and Specker that purports to demonstrate that hidden variable theories for quantum mechanics are excluded, the proof and its significance for the understanding of hidden variable theories and standard quantum mechanics are discussed.

1. INTRODUCTION

Theorems that purport to prove the impossibility of hidden variable reinterpretations of standard quantum mechanics have prompted a variety of responses. It is apparent that their significance for the understanding of the standard theory is far from transparent. One is von Neumann's impossibility proof, published in 1932. Its significance remained obscure until Bell's publication of 1966, and discussion continues.

Another is the hidden variable proof of Kochen and Specker (1967). On the one hand, Bell (1966) sees no good in it and on the other hand Bub (1973a, 1973b, 1974) and others see the proof as providing the ground of and framework for a new "logical" interpretation of the standard theory that radically departs from both the statistical interpretation and the Copenhagen interpretation in its various forms. This paper contains an exposition and critique of this relatively new and novel interpretation. Since the interpretation is based squarely on the Kochen and Specker proof, it is necessary to clarify this proof. It has a significance for our understanding of quantum theory that goes beyond Bell's discussion of the result as a corollary to an important theorem of Gleason's (1957). In the course of this discussion it will be pointed out that advocates of the logical interpretation attribute

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a significance to Gleason's theorem that is at once questionable and illuminating for the understanding of what completeness means for standard quantum mechanics.

The plan of the paper is as follows. First, there is a presentation of the framework of the Kochen and Specker proof. Then follows a discussion of the implications of their criterion for an acceptable hidden variable theory along with a simplified proof sketch of their principal theorem based on the criterion. Next there is a critical discussion of the proof and the grounds for their criterion. I then turn to an exposition and examination of the logical interpretation.

2. THE FRAMEWORK OF THE KOCHEN AND SPECKER PROOF

According to Kochen and Specker, the problem of a hidden variable reinterpretation of quantum mechanics is both "controversial and obscure" because of conflicting results and the lack of any "exact mathematical criterion" by which one could judge the degree of success of various proposed hidden variable theories. The authors propose the missing criterion, argue for its adoption, and demonstrate that it suffices to rule out hidden variable theories for quantum mechanics. Kochen and Specker consider the problem of hidden variables for quantum mechanics within a general framework that they consider suitable for a discussion of classical and quantum mechanics. This framework consists of: a set O of observables, a set S of states (subdivided into pure and mixed), a function p which assigns to each A in O and pure state ψ in S a probability measure on the real line such that "$p_{A\psi}(U)$" denotes the probability that a measurement of observable A for a system in a state ψ yields a value lying in U, where U is a measurable subset of the real line R with respect to $p_{A\psi}$. A hidden variable theory for quantum mechanics may be described within this framework by introducing a phase space Ω of hidden states such that every pure state ψ of quantum mechanics is interpreted as a mixed state of the hidden variable theory, i.e., as a probability measure μ_ψ over Ω. In addition, each observable A of the standard theory is associated with a real-valued function $f_A\colon \Omega \to R$, such that the following *statistical condition* is satisfied:

(I) $\quad p_{A\psi}(U) = \mu_\psi(f_A^{-1}(U))$

or equivalently

$$\mathrm{Exp}_\psi\,(A) = \int_\Omega f_A(r)\, d\mu_\psi(r)$$

This condition says that the probability that the observable A for a system in a state ψ lies in a subset U of R is equal to the probability measure of the points mapped into U by the function f_A on Ω.

Now (I) is a *necessary* condition for an acceptable hidden variable theory. Here "acceptable" means that the hidden variable theory yields precisely the statistical predictions generated by the standard theory. It is usually assumed that the hidden variable distributions rapidly reach an equilibrium distribution and remain there. So it is more exact to say that an acceptable hidden variable theory agrees with the statistical distributions of the standard theory with respect to the equilibrium distributions of the hidden variables. This is not mentioned by Kochen and Specker. They simply demand that the statistical condition be satisfied. They seem more concerned with mathematical criteria than with theoretical considerations that have in fact motivated the construction of hidden variable theories. This is further illustrated by the fact that their statistical criterion rules out *a priori* so-called "local" hidden variable theories. These theories do disagree with the predictions of quantum mechanics. Such theories may be untenable on the basis of experimental evidence but surely are not to be ruled out because they violate a mathematical criterion such as condition (I). They are characterized as unacceptable in this paper *only* in the sense that they are not reinterpretations of the standard theory, i.e., do not agree with its predictions.

Kochen and Specker point out that (I) is not a *sufficient* condition for an acceptable theory. As proof of its insufficiency they cite the fact that the condition is satisfied by trivial constructions of hidden variable theories in which observables appear as *independent* random variables. It is just this feature of the trivial reconstructions that is declared unacceptable. Typically, the observables of a theory are not independent. They are functionally related. As an example they cite the fact that the observable A^2 can be determined by measuring the value of A and squaring the result. Following Kochen and Specker, we may define the function of an observable $g(A)$ for every observable A, state ψ, and Borel function $g: R \rightarrow R$ as

$$(1) \qquad p_{g(A)\psi}(U) = p_{A\psi}(g^{-1}(U))$$

On the assumption that the probability function p determines every observable, we have it that $A = B$ if for every ψ, $p_{A\psi} = p_{B\psi}$ and (1) defines $g(A)$. Through the functional relations existing among the observables of a theory, the set of observables acquires an algebraic structure. Kochen and Specker demand that this structure be preserved in a hidden variable theory, thus eliminating in one swoop the unwanted trivial constructions in which the observables of quantum mechanics are represented by independent random variables on a common phase space. So if $f_A(\lambda) = a_n$ (an eigenvalue of A) and $B = g(A)$, then for state $|a_n\rangle$, B has the value $g(a_n)$; i.e., $f_B(\lambda) = f_{g(A)}(\lambda) = g(a_n) = g(f_A(\lambda))$. Hence, in addition to satisfying condition (I), an acceptable hidden variable theory must also satisfy what I will call a *functional identity* condition:

(II) $f_{g(A)} = g(f_A)$

The Kochen and Specker proof then shows that no hidden variable extension of quantum mechanics satisfies both (I) and (II).

3. THE IMPLICATIONS OF THE FUNCTIONAL IDENTITY CONDITION

Before I raise the question of the reasonableness of the functional identity condition, it is appropriate to review its implications. First, note that a set of observables $\{A_i\}$ ($i \in I$, where I is an index set) is said to be *commeasurable* if there exists an observable B and Borel functions $\{g_i\}$, $i \in I$, such that $A_j = g_j(B)$ for all $j \in I$. By a theorem of Neumark's (1954) each set of commeasurable observables coincides with observables represented by operators that commute pairwise. Now if A_1 and A_2 are commeasurable, $A_1 = g_1(B)$, $A_2 = g_2(B)$, and μ_1 and μ_2 are any real numbers, we may define sums and products of *commeasurable* observables A_1 and A_2 as

$$(2) \qquad \mu_1 A_1 + \mu_2 A_2 = (\mu_1 g_1 + \mu_2 g_2)(B)$$
$$A_1 A_2 = (g_1 g_2)(B)$$

The entire set of observables with sums and products for commeasurable observables defined by (2) is appropriately called a "partial algebra." It is easy to show that condition (II) demands that these partial operations be preserved in a hidden variable extension that associates each observable with a random variable f_A. For sums we have

$$f_{\mu_1 A_1 + \mu_2 A_2} = f_{(\mu_1 g_1 + \mu_2 g_2)(B)} = (\mu_1 g_1 + \mu_2 g_2)(f_B) = \mu_1 g_1(f_B) + \mu_2 g_2(f_B)$$
$$= \mu_1 f_{g_1(B)} + \mu_2 f_{g_2(B)} = \mu_1 f_{A_1} + \mu_2 f_{A_2}$$

And for products we have

$$f_{A_1 A_2} = f_{(g_1 g_2)(B)} = g_1 g_2(f_B) = g_1(f_B) g_2(f_B) = f_{g_1(B)} f_{g_2(B)} = f_{A_1} f_{A_2}$$

Although partial operations are defined only for commuting operators, the *intransitivity* of commutivity means that condition (II) places restrictions on values of hidden variable representatives of *noncommuting* Hilbert space operators. To see this let A, B, and C be operators that commute pairwise with the exception of A and C. Also let B be expressible as a function of both A and C, i.e., $B = g_1(A) = g_2(C)$. Then by condition (II) we have $f_B = f_{g_1(A)} = g_1(f_A)$, and $f_B = f_{g_2(C)} = g_2(f_C)$. Thus condition (II) imposes the restriction $g_1(f_A) = g_2(f_C)$ on the hidden variable representatives of noncommuting A and C. This fact, taken by itself, is hardly grounds for an objection to the condition. What is of interest is that this fact is intimately related to the fact that hidden variable theories cannot satisfy condition (II). To display this crucial connection it suffices to consider the essentials of the proof that condition (II) cannot be satisfied by a hidden variable theory satisfying condition (I).

As we saw, given the definitions of commeasurability, identity of observables, and linear sums and products of commeasurable observables, the set of observables for standard quantum mechanics has the structure of a partial algebra. Condition (II) implies that the partial operations are preserved by the association $A \to f_A$. Now the set R^Ω of functions $f: \Omega \to R$ forms a commutative algebra over R. Kochen and Specker note that condition (II) implies that there is an imbedding of the partial algebra Q of quantum mechanics into the commutative algebra R^Ω, where an imbedding of a partial algebra PA into PA′ is a homomorphism h: PA → PA′ which is one to one into PA′, i.e., a map h such that for all commeasurable $a, b \in$ PA:

$h(a) \# h(b)$ ("#" designates the binary relation of commeasurability)

$$h(\mu a + \lambda b) = \mu h(a) + \lambda h(b)$$

$$h(ab) = h(a)h(b)$$

$$h(1) = 1$$

The imbedding of Q into R^Ω in turn implies the imbedding of the restriction of Q to a partial Boolean algebra of idempotents (projectors) PBA into the restriction of the commutative algebra R^Ω to a Boolean algebra of idempotents BA. Theorem 0 of Kochen and Specker states that a PBA is imbeddable into a BA if and only if for every pair of *distinct* a, $b \in$ PBA there is a homomorphism h: PBA → Z_2 such that $h(a) \neq h(b)$, where Z_2 is the field of two elements {0, 1}.

This requires the existence of a two-valued homomorphism on the partial Boolean algebra of subspaces or the corresponding idempotents. The main theorem then states that there are no two-valued homomorphisms on the PBA of idempotents of a Hilbert space of at least three dimensions. My purposes are served by presenting a proof sketch through an adoption of Bell's (1966) proof of the result as a corollary of Gleason's (1957) theorem.[2]

Let s_1, s_2, and s_3 be three mutually orthogonal one-dimensional sub-

[2] The proof sketch presented here is also informed by versions of the proof due to Belinfante (1973) and Bub (1974).

spaces of a three-dimensional Hilbert space, and let h be a homomorphism from the partial Boolean subalgebra into Z_2. Then we have

$$h(s_1) \cup h(s_2) \cup h(s_3) = h(s_1 \cup s_2 \cup s_3) = 1$$

and

$$h(s_i) \cap h(s_j) = h(s_i \cap s_j) = h(0) = 0 \qquad \text{for } 1 \leqslant i \neq j \leqslant 3$$

So for orthogonal triples of lines replaced by orthogonal triads ϕ_i of unit length, h defines a map from the triads onto $\{0, 1\}$ such that

(a) $h(\phi_i) = 1$ or 0 $(i = 1, 2, 3)$
(b) $\sum_i h(\phi_i) = 1$
(c) If $h(\phi_i) = 1$, then $h = 0$ for the remaining directions normal to ϕ_i

Now assume that $h(\phi_1) = h(\phi_2) = 0$. It follows that for ϕ_3 normal to ϕ_1 and ϕ_2, $h(\phi_3) = 1$. Select any vector $(a\phi_1 + b\phi_2)$ in the plane spanned by ϕ_1 and ϕ_2. $(a\phi_1 + b\phi_2)$ is normal to ϕ_3 and so $h(a\phi_1 + b\phi_2) = 0$. Thus we have

(d) If $h(\phi_1) = h(\phi_2) = 0$, then $h = 0$ for all other directions in the plane of ϕ_1 and ϕ_2.

By repeated applications of (c) and (d) the theorem is proved and (d) is a necessary condition for the success of the proof. Note that (d) implies that for *any* rotation of the triad about ϕ_3 normal to ϕ_1 and ϕ_2, where $h(\phi_1) = h(\phi_2) = 0$, $h(\phi_3) = 1$. Thus the value of $h(\phi_3)$ is invariant under rotation. This means that the value of $h(\phi_3)$ does not depend on the other members of the orthonormal set, say ϕ_1', ϕ_2' to which it belongs. That is to say, $h(\phi_3) = 1$ given that $h(\phi_1) = h(\phi_2) = 0$, regardless of whether ϕ_3 belongs to $\{\phi_1, \phi_2, \phi_3\}$ or to $\{\phi_1', \phi_2', \phi_3\}$. I will refer to this as "basis insensitivity." Clearly, if the above is denied, the demand for the imbedding forced by condition (II) cannot be met, for the imbedding implies the invariance of the value of ϕ_3 under rotation about itself. It is this demand that is objected to by Bell as follows:

> How did it come about that (B)[3] was a consequence of assumptions in which only commuting operators were explicitly mentioned? The danger was not in fact in the explicit, but in the implicit assumptions. It was tacitly assumed that measurement of an observable must yield the same value independently of what other measurements may be made simultaneously (Bell, 1966).

[3] Bell's (B) is equivalent to my (d).

Bell refers to the demand of basis insensitivity as a "tacit assumption." As we have seen, this demand is tacit in the sense that it is *implied* by the requirement for the homomorphism forced by condition (II). This went unnoticed by me for some time, and perhaps it has not been clear to others.

Now, it is important to note that basis insensitivity, without which the proof cannot succeed, forces relations between random variables associated with noncommuting operators. To see this consider the projection operator associated with the subspaces concerned, i.e., consider $P(\phi_1)$, $P(\phi_2)$, $P(\phi_3)$, and $P(\phi')_1$, where $P(\phi_1') = P(a\phi_1 + b\phi_2)$ and neither a nor b is equal to zero. Note that $P(\phi_1')$ is defined by a rotation about ϕ_3. We have

$P(\phi_1) \# P(\phi_3)$ ("#" denotes the binary relation of commeasurability)

and

$$P(\phi_3) \# P(\phi_1')$$

but *not*

$P(\phi_1) \# P(\phi_1')$ (the intransitivity of commeasurability)

For quantum mechanics we have $\text{Exp}\, P(\phi_1) = \text{Exp}\, P(\phi_2) = 0$ implies $\text{Exp}\, P(\phi_1') = \text{Exp}\, P(a\phi_1 + b\phi_2) = 0$. But for hidden states the expectation values correspond to eigenvalues. Thus a relation is forced between the value of $f_{P(\phi_1)}$ and $f_{P(\phi'_1)}$ where $P(\phi_1)$ and $P(\phi_1')$ do *not* commute. We see now that this is not an independent fact. It follows from premise (d), which is necessary for the Kochen and Specker proof and is forced by the demands of the functional identity condition.

4. THE UNREASONABLENESS OF THE FUNCTIONAL IDENTITY CONDITION

We have seen that condition (II) places demands on hidden variable extensions of quantum mechanics that cannot be satisfied. It is true that the condition provides an exact mathematical criterion for a hidden variable theory. The question is whether it is reasonable. Kochen and Specker suggest two grounds for adopting the condition.

After defining the notion of a function of an observable, the authors say:

> Thus the measurement of a function $g(A)$ of an observable A is *independent of the theory* considered—one merely writes $g(a)$ for the value of $g(A)$ if a is the measured value of A. The set of observables of a theory thereby acquires an algebraic structure, and the introduction of hidden variables into a theory should preserve this structure (Kochen and Specker, 1967). (Italics are mine.)

The argument is intended to show that the algebraic structure of the observables of a theory is independent of the theory under consideration. It is supposed that this structure is given empirically through the measurement of observables and calculations of other observables functionally related to them. Now clearly, the functional relations in question are not independent of the theory considered. In particular, the algebraic structure is generated under the *assumption* that every observable is determined by the probability function p, i.e., that if $p_{A\psi} = p_{B\psi}$ for every state ψ, then $A = B$. This is hardly theoretically neutral. It is certainly not given empirically, nor can it qualify as a necessity of thought. It is just as natural to question the idea of identity of observables based on statistical agreement. Indeed this is just what is emphasized in current hidden variable theories. Of course, it is a legitimate assumption to adopt for constructing a statistical theory such as standard quantum mechanics. The point is that it is not an assumption that one is bound to preserve in a hidden variable extension. So we may conclude that the algebraic structure of the observables of a theory is not theory neutral, but depends on assumptions that go far beyond what is measured.

Furthermore, the demand that we should have $(f_A + f_B)(\lambda) = f_A(\lambda) + f_B(\lambda)$ for commuting A and B is also questionable. Of course, it holds in the case where the state of the system is a common eigenstate. When this is not so, quantum mechanics is silent on the question as to whether $A + B$ represents the sum of the corresponding operators. It is true that in a simultaneous measurement of A, B, and $A + B$ on a system in a superposition ψ of eigenstates of A and B we obtain individual results in accord with $\text{Exp}_\psi A + \text{Exp}_\psi B = \text{Exp}_\psi (A + B)$. But this gives no functional relations between *individual* measurements. Since expectation values for hidden states correspond to eigenvalues, it is hardly reasonable to demand in *general* that $(f_A + f_B)(\lambda) = f_A(\lambda) + f_B(\lambda)$. The point is that the rule for sums of commuting operators is an *hypothesis* that works well for a statistical theory such as standard quantum mechanics and it plays a key role in determining the algebraic structure of the theory. It is *not* independent of the theory and hence the algebraic structure is not independent of the theory as Kochen and Specker suppose. Moreover, this rule cannot reasonably be used to impose restrictions on the values of individual measurements as predicted by hidden variable theories because in general the rule can only impose relations between expectation values and has nothing to say about eigenvalues found in single measurements with the sole exception of a system in a common eigenstate with respect to A and B.

The alleged theory independence of the algebra of operators is not the only reason given in support of condition (II). The authors further support condition (II) by claiming that, while hidden variable theories have explicitly demanded *only* that condition (I) be satisfied, condition (I) is not sufficient since it permits trivial constructions with the common property that random variables associated with quantum mechanical operators are independent. This is taken as indicative of the need for condition (II), which obviously rules out the trivial constructions.

But condition (I) is not the only criterion appealed to in hidden variable constructions. For example, Bohm had insisted that the hidden variables depend on both the state and the measuring device. He writes

> However, in our suggested new interpretation of the theory, the so called "observables" are ... not properties belonging to the observed system alone, but instead potentialities whose precise development depends just as much on the observing apparatus as on the observed system. ... the distribution of hidden parameters varies in accordance with the differently mutually exclusive experimental arrangements of matter that must be used in making different kinds of measurements (Bohm, 1952).

Now this requirement rules out condition (II), and the resulting hidden variable theory is nontrivial. It is instructive to see why condition (II) is ruled out by Bohm's hypothesis concerning the nature of observables. In the proof sketch of the Kochen and Specker theorem we saw that assumption (d) implies that the value (1 or 0) assigned to a statement asserting that an observable has a certain value is functionally independent of the particular eigenbasis chosen. Bohm's condition contradicts this by demanding that the value of such a statement functionally depends on the particular eigenbasis defined by the measuring apparatus. Thus the Kochen and Specker theorem is squarely based on a condition, namely condition (II), that is *incompatible* with a principle hypothesis of a hidden variable theory. In view of this, the Kochen and Specker theorem does not provide a reasonable mathematical criterion for an acceptable hidden variable reinterpretation of quantum mechanics.

The fact that the Bohm condition [in contrast with condition (II)] demands that an observable is assigned a value relative to the specification of a particular eigenbasis means that the hidden variable theory does not specify simultaneous values for noncommuting operators, but only predicts what will be found relative to a measurement sufficiently precise to define an eigenbasis. Thus any hidden variable extension of the standard theory must satisfy Bohr's dictum that the entire experimental arrangement must be taken into account. In short, observables are not properties of systems revealed by measurement. Of course, this is hardly a novel view. What is of interest is that Bohr's dictum must be satisfied in any hidden variable extension of the quantum theory. It follows that complementarity is maintained.

It is true that the partial algebra of operators of the standard theory is not conserved in the Bohm and Bohm–Bub hidden variable extensions. But note that in standard quantum mechanics the partial algebra plays the role of assuring complementarity, whereas in the Bohm–Bub hidden variable theory (Bohm and Bub, 1966), complementarity is assured by the algorithm for generating individual predictions. While it is necessary to retain complementarity in an acceptable hidden variable theory, it need not be retained through the preservation of the algebra of the operators that is peculiar to the formulation of the standard theory.

5. THE KOCHEN AND SPECKER THEOREM AND THE LOGICAL INTERPRETATION

My discussion of the Kochen and Specker theorem would appear to be at odds with those investigators who use the result as the foundation of the so-called logical interpretation of the quantum theory. There are several reasons why this interpretation deserves consideration. First, it contrasts sharply with the two most widely held interpretations—the Copenhagen and the statistical interpretations. Its proponents are as confident of its merits as they are critical of what they judge to be the shortcomings of the standard interpretations. Secondly, the principal advocates of the interpretation make such heavy and confident use of sophisticated mathematical machinery in their exposition of the view as to inspire curiosity about the actual basis for the novel claims they make. Also, the very fact that Bub, who is presently a leading proponent of the logical interpretation, formerly defended the hidden variable approach, is good reason to be curious. Finally, the fact that the interpretation is so closely tied to the Kochen and Specker proof challenges the more "orthodox" understanding of that proof that I have expounded and expanded on in the preceding sections.

As a matter of history, the logical interpretation has two sources. The first track begins with Specker's 1960 paper, "Die Logik nicht gleichzeitig entschiedbarer Aussagen." This was followed by two papers by Kochen and Specker exploring the logical calculus of partial propositional functions (1965a, b). The final contribution of these authors is their study of hidden variables, of which Section VII is devoted to "The Logic of Quantum Mechanics" (1967). The second track begins with Finkelstein's (1962–63) revival of the work of Birkhoff and von Neumann (1936). This was taken up by Putnam (1969), who in turn converted Bub and Demopoulos (1974) to the logical interpretation. Bub and Demopoulos (unlike Putnam) concentrated their efforts on a formal presentation of the interpretation based squarely on the work of Kochen and Specker. Thus do the two tracks of development merge.[4]

The reader should be warned not to confuse the logical interpretation with other approaches to question of interpretation and foundational studies that are loosely described as concerned with "quantum logic." In particular, the logical interpretation is completely distinct from the earlier work of Birkhoff and von Neumann, the later work of Finkelstein, and views expressed by Jauch, Piron, and Strauss, to name but a few. Indeed the whole field of quantum logic represents a family of diverse studies and views. Jammer (1974) has sorted matters out to some degree but much remains to be clarified and it clearly deserves a separate study. The concern of this paper is strictly limited to the logical interpretation as expounded by its leading proponents, Bub and Demopoulos. I will proceed by listing the essential features of the interpretation and then consider the basis for the distinctive elements of the interpretation.

The main components of the logical interpretation are as follows:

L1. The interpretation problem consists in specifying the nature of the transition from classical mechanics to quantum mechanics. This is characterized as a transition from the Boolean event structure of classical mechanics to the partial Boolean event structure of quantum mechanics.

L2. Standard nonrelativistic quantum mechanics is both a principle and a complete theory.[5]

L3. A correct understanding of the role of the event structure of quantum mechanics shows that: (a) quantum mechanical systems have properties in precisely the same sense as classical systems are said to have properties. (This stands in sharp contrast to a wholistic account wherein properties of systems are defined relative to an experimental context.) (b) There are states (represented by unit rays or one-dimensional subspaces) that do not determine whether a property (represented by a projection operator) holds or fails to hold. However, it is always determinate whether a property holds or fails to hold of a system, for, at a given time, a system has many states—enough to determine whether any property holds. (c) Basic propositions of quantum mechanics make assertions about the properties of individual systems, not ensembles of them.

[4] The data for the first track come from the writings of its participants. The facts about the second track are known to me through conversations with Finkelstein, Putnam, Bub, and Demopoulos.

[5] The notions of a principle and complete theory are defined in the discussion that follows.

L4. The interpretation has the consequence that there is no measurement problem. This argues for its superiority over competitive interpretations that do generate a measurement problem.

I now turn to clarification, analysis, and comment. The expressions "logical structure," "event structure," and "possibility structure" are used interchangeably in expositions of the interpretation. These expressions refer to the partial Boolean algebra of the projection operators (or equivalently, the algebra of the associated one-dimensional subspaces) of a separable complex Hilbert space and to the logic that may be constructed from the algebra. Since an *event* is defined as a one-dimensional subspace, the expression "event structure" is appropriate. The logic associated with the Boolean algebra makes use of sentences of the form "The value of physical magnitude A lies in U." These sentences correspond one to one with the subspaces and the associated projectors.

There is no objection to characterizing the interpretation problem as a specification of the nature of the transition from classical to quantum mechanics. Moreover, it is true that it is possible to describe this transition in terms of the mathematical framework of both theories by noting that whereas classical mechanics has the event structure (i.e., phase space structure) of a Boolean algebra, quantum mechanics has the structure of a partial Boolean algebra. It is also true that the two structures are remarkably different and this difference can be correctly described by noting that the latter is not imbeddable in the former. (See Section 3 above.) All of this is mathematically correct. However, the question of the significance of this specification of the transition for the understanding of the theory is by no means settled by its mathematical description. As we shall see, the answer to that question depends on other considerations. But first I take up the claim that quantum mechanics is a principle and complete theory. (See L2 above.)

Both classical and quantum mechanics are said to be "principle" theories (Bub and Demopoulos, 1974). This is short for the claim that both theories are "theories of logical structure," where "the logical structure of a theory is understood as imposing the most general constraints on the occurrence and nonoccurrence of events" (Bub and Demopoulos, 1974). Thus logical structure is given a realistic as opposed to a syntactical or semantical interpretation. This means that logical structure is postulated as an abstract but real feature of the world that determines how events may occur, or, to use Bub's picturesque phrase, "how properties hang together" (1974). It is undeniably true that the set of projection operators of quantum mechanics form a partial Boolean algebra. It is also true that a partial Boolean algebra can be translated into a partial Boolean logic just as a Boolean algebra can be translated into a Boolean (i.e., classical) logic. Any arguments against the logical interpretation to the effect that the partial Boolean logic is suspect as a logic are simply bad arguments. But this hardly amounts to an endorsement of the logical interpretation. The question that naturally arises at this point is whether there is any reason to accept the assertion that this logic (or algebra) should be "understood as imposing the most general constraints on the occurrence and nonoccurrence of events." The straightforward way of understanding the logical structure of quantum mechanical observables is to understand it as defined *via* the set of statistical states. Indeed, this is how Kochen and Specker proceed. Equivalence of magnitudes, functional relationships, and commeasurability are defined *via* the set of statistical states. The theoretical motivation for this procedure consists in the fact that there is no reason for treating magnitudes of the theory as equivalent aside from the fact that they have identical probability distributions for every statistical state. The question of the significance of this logic for the understanding of quantum mechanics has no *unique* answer. However, it does have a reasonable answer that does *not* lend support to the claim that quantum mechanics is a theory of logical structure in the sense specified by the logical interpretation. It is natural to understand the partial Boolean logic of properties associated with projectors as a logic of complementary dispositional properties, the equivalence of which reflects nothing but the identity of the statistical distribution of the associated magnitudes. The use of the term "natural" here is justified by two considerations. The replacement of a classical (i.e., Boolean) logic by a partial Boolean logic is a necessary and sufficient condition for correcting the defect that meaningless compound sentences can be formed from meaningful sentences if the predicates concerned are dispositional.[6] To my knowledge, Martin Strauss

[6] An example of a dispositional (or reactive) predicate is "soluble in water." Strauss (1972) points out that such predicates can only be partially rather than explicitly defined. For example, "x is soluble in water" is defined as "x is in water and x is soluble in water if and only if x is dissolving in water." Note that such predicates cannot be defined independently of the conditions under which the corresponding properties are observed. If, following Strauss, we represent two different dispositional predicates by X and Y and their defining conditions by E_1 and E_2 and by E_3 and E_4, respectively, the above definition may be written: E_1 & X if and only if E_2. Some other dispositional predicate Y may be defined as: E_3 & Y if and only if E_4. The conjunction of X and Y is then defined as (E_1 & E_3) & (X & Y) if and only if (E_2 & E_4). But the conditions E_1 & E_3 and the conditions E_2 & E_4 may exclude one another. Hence, the conjunction X & Y may be undefined, even though X and Y are separately definable. For an informative treatment of the dispositional character of the magnitudes of quantum mechanics, the interested reader should refer to "Do Quanta Need a New Logic?" by John Stachel, to appear in the University of Pittsburgh Series in the *Philosophy of Science*.

is the first and only investigator to point this out (1972). Unfortunately, his observation has been ignored.[7]

Secondly, maximally consistent sets of sentences, each of which asserts that a system has a specific property at a common time, belong to different Boolean subalgebras of a partial Boolean algebra and correspond to sets of properties that are ordinarily called "complementary" following Bohr. Thus a partial Boolean logic is a logic of complementary dispositional properties. On this view the logic provides a consistent and logically rigorous way for talking about the properties of individual systems in the spirit of Bohr. There is no claim that the logic has the ontological status of being an element of reality that constrains events.

We see then that there is a sharp difference between the two approaches to quantum logic. The approach outlined above might be called "semantical", to mark the fact that the logic provides a way of talking about individual measurements that is consistent with the statistical predictions of the theory. On the other hand, the quantum logical interpretation is called "realistic" by its proponents to stress that the logical structure is a "feature" of the world that constrains the occurrence of events. The former is helpful in so far as it is a logical framework for an exact explication of the central tenets of the Copenhagen interpretation. The latter involves the denial of that interpretation in its treatment of properties, measurement, and its postulation of logical structure as an element of reality.

What are the reasons given for the postulation of logical structure in the sense specified by the logical interpretation? The first reason given by Bub is that it is necessary to postulate that the observables of quantum mechanics have the structure of a partial algebra if we are to understand the Kochen and Specker proof properly. He says

> Only by *assuming* that equivalence in the partial algebra of magnitudes is not merely statistical can we claim the Kochen and Specker theorem as a proof of the impossibility of representing the statistical states of the theory as measures on a classical probability space (Bub, 1974).

Bub therefore objects to defining commeasurability, functional relationship, and, in particular, equivalence in the algebra of magnitudes in terms of the set of statistical states. And he accuses Kochen and Specker of contributing to a "misunderstanding" of their proof by employing such a definition. The misunderstanding in question is the view that the proof only shows that there is no "imbedding of a partial algebra of quantum

[7] Credit goes to Strauss as the first to recognize the importance of partial Boolean logic for studying problems of interpretation of quantum mechanics.

mechanical magnitudes into a commutative algebra of random variables on a probability space." On this view one might object to the proof in the manner of the preceding sections of this paper. That is to say, there is simply no compelling reason to treat random variables corresponding to equivalent magnitudes as equivalent in a hidden variable reinterpretation of the theory when equivalence is determined by agreement in probability distributions. One might well postulate that the observables, represented by different random variables in a hidden variable theory, are statistically equivalent only for probability distributions corresponding to the statistical states of standard quantum mechanics. Clearly, this reasoning undercuts the force of condition (II) of the proof. Thus, Bub insists that the partial algebra is fundamental—that it does not merely reflect the statistical distributions of the standard theory. Hence, he postulates the logical structure outright without definition of equivalence of observables *via* statistical distributions.

Given this, we are told

> The contribution of Kochen and Specker lies in showing that the problem of hidden variables is not that of fitting a theory—i.e., a class of event structures—to a statistics. ... Rather the problem concerns the kind of statistics definable on a given class of event structures. The Kochen and Specker proof is a demonstration that the statistics definable on the event structures of quantum mechanics is not representable by probability measures on a classical probability space (Bub, 1974).

Four brief comments are in order. First: The argument for the postulation of the logical structure of the magnitudes amounts to nothing more than this: assume condition (II) and the Kochen and Specker proof excludes hidden variables. We have seen that condition (II) is unreasonable. Second: There is absolutely no theoretical motivation for treating quantum mechanical magnitudes as equivalent aside from agreement of statistical distributions. Third: To be sure, the problem of hidden variables is not merely the fitting of a theory to a statistics. Bub is right in pointing out that this can be done in any number of trivial ways. But we need not assume condition (II). Whatever problems there are with hidden variable theories concerns their theoretical completeness and testable experimental consequences. These are problems of physics, not of *a priori* conditions such as condition (II). Fourth: Even if one should postulate the structure of the magnitudes independent of statistical distributions, this is no reason for assuming that this structure is a feature of reality that imposes the "most general constraints on the occurrence and nonoccurrence of events." At most it mirrors functional relationships among the magnitudes of the standard theory and provides a rigorous way of reporting individual measurements that is bound not to run into problems because the logic is a logic of complementary dispositional properties.

The second reason given by Bub for postulating the logical structure as independent of the statistical states is that the "completeness problem makes sense only with respect to a given class of structures" (1974). The completeness to which Bub refers is established by Gleason's theorem (1957). This is the very theorem from which the Kochen and Specker result follows as a corollary. Gleason's result is not usually thought of as a completeness result because the problem he addresses is the foundational problem of establishing von Neumann's statistical algorithm for quantum mechanics [$\mu(\kappa) = \mathrm{Tr}\,(WP)$, where W is a statistical operator and P the projector onto κ] on fewer and less suspect assumptions than those employed by von Neumann. Now it so happens that Gleason's theorem can be correctly understood as demonstrating a kind of *statistical* completeness. But as we shall see, this concept of completeness has nothing to do with the completeness Bohr attributed to quantum mechanics nor with the element of completeness focused on by hidden variable theorists such as Bohm and even Bub when he previously championed hidden variable theories.

Gleason's theorem can be understood as showing "that the quantum algorithm generates all possible generalized probability measures on the partial Boolean algebra of propositions of a quantum mechanical system" (Bub, 1974). Note that completeness is defined *relative* to a "class of event structures," that can be alternatively described as the partial Boolean algebra of the closed linear subspaces of a Hilbert space or the associated projectors or corresponding propositions. Bub's point is that the completeness of the statistics for the partial algebra shows that the partial algebra is fundamental and thus that any hidden variable reinterpretation is "uninteresting if the structure of the magnitudes is not preserved" (1974). In other words, Gleason's proof guarantees completeness relative to a class of structures that Kochen and Specker show to be fundamental. Thus Bub says

> However, the significance of Gleason's theorem for the completeness of quantum mechanics is only fully brought out by Kochen and Specker's notion of a partial algebra, which completely clarifies the sense in which the Boolean event structures of classical mechanics are generalized by quantum mechanics (1974).

Now the statistical completeness established by Gleason's theorem has no bearing on the question of completeness addressed by Bohr and Bohm. Bohr insisted that quantum mechanics was complete in the sense that there could not be any other meaningful physical parameters that enter into the specification of the state of a system. Bohm denies this. Clearly, a hidden variable theory such as Bohm's is incompatible with Bohr's assertion of completeness but compatible with the *statistical* completeness established by Gleason. It is one thing to assert that all probability measures are given by the quantum mechanical algorithm relative to the closed linear subspaces of a Hilbert space and quite another to assert that all possible empirically relevant information is included in the specification of the state in standard quantum mechanics.

The proponents of the logical interpretation have confused two entirely different concepts of completeness, namely, statistical completeness and completeness of the specification of the state. In particular, the fact that standard quantum mechanics is statistically complete does not warrant the conclusion that "equivalence in the algebra of the projectors is not merely statistical" (Bub, 1974). The remainder of my remarks about the logical interpretation are addressed to items L3 and L4 as specified above.

In his remarks on the Kochen and Specker proof viewed as a consequence of Gleason's theorem, Bell says:

> In fact it will be seen that these demonstrations require from the hypothetical dispersion free states, not only that appropriate ensembles thereof should have all measurable properties of quantum mechanical states, but certain other properties as well. These additional demands appear quite reasonable when results of measurement are loosely identified with properties of isolated systems (1966).

The logical interpretation endorses just what Bell points out as a weakness of hidden variable proofs. It treats properties as properties that systems *have* in the classical sense. The question is whether this account holds up. Bell cites Bohr's point that it is impossible to maintain a "sharp distinction between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear" as evidence against the assumption that quantum mechanical systems have properties in the classical sense. But one not already convinced of Bohr's point must examine the claim of the logical interpretation on its own merits.

It is well known that quantum mechanical properties (strictly speaking, equivalence classes of properties) can be represented by projection operators. The construction is due to von Neumann (1955). Although the logical interpretation employs this construction, agreement with von Neumann is confined to the construction. The interpretation is radically different.

The claim of the logical interpretation is that a quantum mechanical system S has all its properties in just the sense in which a classical system is said to have its properties and that "the properties of S which obtain simultaneously include *incompatible*[8] properties" (Demopoulos, 1976; Bub, 1974, 1976). I will refer to this claim as *the property postulate*. Clearly, the property postulate represents a bold departure from all previous interpretations of the theory. In essence, the argument for the property postulate purports to show that the treatment of properties in quantum mechanics is "exactly analogous" to the classical case. I now turn to the details of the argument.

First, the conditions under which a classical and quantum system has a property are defined in analogous ways. In classical mechanics the idempotent magnitudes representing properties are the characteristic functions χ_Γ [where $\Gamma \equiv f_A^{-1}(U)$, $f_A\colon \Omega \to R$, $U \subseteq R$, R is the real line; and Ω is a subset of $6N$-dimensional Euclidean space). In quantum mechanics the idempotents are the projection operators P. For classical mechanics, a property represented by χ_Γ holds for a system S "if and only if S is in a state $\mathbf{w}$ such that $\chi_\Gamma(\mathbf{w}) = 1$" (Demopoulos, 1976). Here it is crucial to understand that, for the logical interpretation, the concept of a state in *both* classical and quantum mechanics is *not* a statistical concept (Bub, 1976; Demopoulos, 1976). A classical state is represented by a point $\mathbf{w}$ in Ω (a subset of $6N$-dimensional Euclidean space) and it is associated with a pure *statistical* state, namely, the measure on the field of subsets of Ω. Analogously, a quantum state is represented by a (unit) ray $\mathbf{K}$ (or one-dimensional subspace) in a complex separable Hilbert space $\mathbf{H}$ and it is associated with a pure *statistical* state, namely, a measure on the closed linear subspaces of $\mathbf{H}$. Now with reference to *this* concept of state, the logical interpretation defines the holding of a property for a quantum mechanical system in strict analogy with the classical case as follows: A property represented by a projection operator P holds for a quantum mechanical system S if and only if S is in a state $\mathbf{K}$ such that $P(\mathbf{K}) = 1$ (Demopoulos, 1976; Bub, 1976).

Now the fact that the logical interpretation gives strictly analogous definitions for the holding of a property in classical and quantum mechanics hardly suffices to establish the property postulate. Indeed, an elementary objection appears to apply. In the quantum case (unlike the classical case) there are states $\mathbf{K}$ such that they do not determine whether or not a given property P holds. This objection focuses on what is central to the logical interpretation:

> The determinateness of the holding of P is completely independent of whether or not the holding of P is determined by every state of S. I regard this claim as central to the logical interpretation of quantum mechanics. This is obscure if it is

[8] The term "incompatible" is used in the logical interpretation to refer to properties that are not commeasurable.

> assumed that P is a property of a state rather than a property of S and that S has only a single state **K** (Demopoulos, 1976; cf. Bub, 1976).

The claim is that although S is characterized by a single statistical state, S has enough (nonstatistical) states to determine the truth or falsity of each sentence attributing a property to S. This being so, "the properties of S that obtain simultaneously include incompatible properties."

Now a sentence of the form "S has P_1 & S has P_2", where P_1 and P_2 are incompatible properties, is *not defined* in the partial Boolean logic of quantum mechanical propositions. How then can the properties of S that obtain simultaneously include incompatible properties? At most, one could claim that S is characterized by one set of compatible (i.e., commeasurable) properties. In this case there is neither need nor justification for an interpretative hypothesis affirming that at a given time a system has a plurality of (nonstatistical) states.

The answer given by the logical interpretation is that the absence of a simultaneous truth value assignment to "S has P_1" and "S has P_2," where P_1 and P_2 are incompatible properties, does not imply that P_1 and P_2 cannot obtain simultaneously:

> That simultaneous truth value assignments do not exist is a fact about the structure of N (the partial Boolean logic) which has nothing to do with what occurs simultaneously (Demopoulos, 1976).

If the argument is meant to show that the properties of S that obtain simultaneously include incompatible properties, then the argument is invalid. This conclusion does not follow from the premise that the absence of simultaneous truth value assignments has nothing to do with what obtains simultaneously. All that follows is that one cannot draw any conclusions about whether or not incompatible properties obtain simultaneously.

On a more generous reading the argument may be taken to show that it is *possible* that incompatible properties obtain simultaneously. That is to say, this possibility is not ruled out by the fact that a statement to this effect is not defined in the partial Boolean logic of quantum propositions. On this reading the property postulate is not required by the partial Boolean logic of properties. So what is said to be central to the logical interpretation—namely, the property postulate—is revealed to be an assumption not implied by the quantum logic of properties. At best, it is an article of faith totally unmotivated by theory or experiment. Moreover, it is difficult to make sense of the idea that incompatible properties obtain simultaneously since this idea is not expressible by any true (or false) sentence in the partial

Boolean logic of properties. It is far better to say, "Where one cannot assign truth values, let one be silent."

By the same token it is difficult to make sense of the notion that, at a given time, a quantum mechanical system "has many states"—sufficient to establish the truth or falsity of every statement asserting that the system has a property. Of course, there are sufficient (unit) rays to *define* the truth or falsity of each such statement, but this does not mean that a system can be *characterized* as being in these states simultaneously. Clearly, such a characterization would violate the uncertainty relations.

An even more serious objection concerns the assumption that every quantum system is in a state represented by a unit ray. This concept of state ignores the role of mixtures in quantum theory and the implications of the concept of a mixture for the possibility of an individual state concept that applies universally. It is the contrast of quantum mechanics with classical statistical mechanics that renders the state concept of the quantum logical interpretation unacceptable. Although *classical* ensembles are often characterized as mixtures, it is legitimate to assume that an individual state concept applies universally and to postulate that a system is always in a *pure* state represented by a point $\mathbf{w}$ of phase space. Contrary to the logical interpretation, there is no analog for this in quantum mechanics. In classical physics mixed ensembles can be reduced to a *unique* set of pure subensembles thus making possible the interpretation of any ensemble as a collection of individual systems in definite states. In quantum mechanics mixed ensembles cannot be reduced to a unique set of pure subensembles. A plurality of reductions is possible (von Neumann, 1955). Thus the assignment of a state vector to individual quantum system is ambiguous and leads to paradoxes (Park, 1968).

Park (1968) points out a second crucial difference between classical and quantum ensembles. When a pure classical ensemble interacts with a second pure classical ensemble, it remains pure during its temporal development. Thus a classical system can be meaningfully assigned a sequence of states throughout its temporal development. By way of contrast, a closed pure quantum ensemble does remain homogeneous throughout its temporal development but upon interaction with a second pure ensemble is converted into a mixture. This is often referred to as "quantum entanglement."[9]

[9] "Classical entanglement" differs from quantum entanglement in two ways. First, classical entanglement occurs when a pure ensemble E_1 is converted into a mixture by interaction with a *mixed* ensemble E_2, whereas quantum entanglement occurs under conditions of maximal homogeneity, i.e., both ensembles are initially pure. Secondly, classical entanglement is not problematic for an individual state concept since the E_1 "ensemble" consists of a "distribution of *different kinds of systems*, characterized by *different* Hamiltonians" (Park, 1968). Hence the inhomogeneity of E_2 is reflected in the Hamiltonians characterizing E_1. Clearly there is no such explanation for quantum entanglement.

Since initially pure quantum ensembles lose their homogeneity through entanglement, one cannot always give an independent account of the temporal development of initially pure quantum ensembles. So it is not always possible to assign a temporal sequence of state vectors to a quantum system. It is interesting to note that these two objections also rule out the idea that a quantum system always has a complete set of *compatible* properties. This is a weaker thesis than the property postulate and its accompanying individual state concept. Thus the nature of quantum mixtures places severe restrictions on the class of permissible interpretations.

The final feature of the logical interpretation to be considered is the claim that there is no measurement problem (Bub, 1974, 1976; Demopoulos, 1976). Many physicists share the opinion that the measurement problem is a pseudoproblem, but the reasons for this opinion differ sharply from the reasons offered by the proponents of the logical interpretation. The difference may be exhibited as follows.

Consider a microsystem S and a measuring instrument M that measures a magnitude A of S. For my purposes it suffices to exhibit an elementary description of the measurement process. Let us assume that there is an interaction between S and M such that the state of the initial composite system $S + M$ is transformed by a unitary transformation U that establishes a unique correspondence between the eigenvalues a_i of A and b_i of a pointer reading magnitude B of M. If the initial state of the microsystem S is $|\psi\rangle = \sum_i \langle\alpha_i|\psi\rangle|\alpha_i\rangle$ and the initial state of the apparatus is ϕ, the measure interaction results in the transition

$$\text{(M-1)} \qquad U|\psi\rangle|\phi\rangle = |\psi'\rangle = \sum_i \langle\alpha_i|\psi\rangle|\alpha_i\rangle|\beta_i\rangle$$

where α_i (β_i) is an eigenvector corresponding to the eigenvalue a_i (b_i).

(M-1) may be understood as meaning that if the experiment were repeated many times with identical state preparation conditions for S and M, the relative frequency of b_i would be found to be $|\langle\alpha_i|\psi\rangle|^2$. This is identical with the probability of S being found to have the value a_i of magnitude A. The reader will recognize this as characteristic of the statistical interpretation of the theory in which the state vector is understood as characterizing an ensemble of similarly prepared systems. This reading of (M-1) reflects the verifiable statistical predictions of the theory and generates no difficulties for the notion of measurement.

However, if one assumes that the state vector completely characterizes an individual system and that the measuring apparatus is to be analyzed quantum mechanically, the interpretation of (M-1) then becomes problematic, for (M-1) is a coherent superposition of vectors corresponding to distinct (pointer) values b_i. This means that the pointer is in a superposition and does not have a definite reading at any time. There are an enormous variety of proposals of how to solve or avoid this problem. The logical interpretation offers the solution that the measurement problem can be avoided by assuming that a system has an atomic property even though it is represented by a vector that is *not* in the one-dimensional subspace that is in the range of the projection operator representing the property (Bub, 1974, 1976; Demopoulos, 1976). On this assumption the composite system $S + M$ represented by $|\psi'\rangle = \sum_i \langle\alpha_i|\psi\rangle|\alpha_i\rangle|\beta_i\rangle$ really does have a property corresponding to a value b_i of B and a_i of A for some i. Measurement merely reveals what these properties are (Bub, 1974). Thus the logical interpretation appeals to the property postulate to avoid problems of measurement. If my reasons (given above) for rejecting the property postulate are correct, the quantum logical interpretation fails to avoid interpretative problems concerned with measurement.

It is instructive to note that the proponents of the logical interpretation have failed to appreciate that although the ensemble of composite systems $S + M$ remains homogeneous throughout its temporal development, the *subsystems* S and M are in general converted into mixtures. For this reason there is no theoretical basis for the claim that the observable A has a definite value when the ensemble of composite systems is represented by the vector $|\psi'\rangle$.

6. CONCLUDING REMARKS

The purpose of these remarks is to obviate any misunderstandings that may result from my analysis of the Kochen and Specker proof and the logical interpretation of quantum mechanics. First, the paper is not to be taken as an endorsement of any hidden variable theory. However, the continued interest in hidden variable theories not ruled out by present experiments is fully consistent with the conclusions of this study. They are best described as explorations of intuitive ideas that may illuminate the "quality of quantum interconnectedness" (Bohm and Hiley, 1975). Strong elements of nonlinearity are introduced, and this sacrifices the simplicity of the standard theory. But this cannot justify a rejection of these explorations. Simplicity—a concept that is notoriously difficult to define—is no guarantee of truth.

Secondly, my reference to the partial Boolean algebra of dispositional complementary properties is not to be understood as an endorsement of an individual interpretation, i.e., an interpretation that postulates that the standard theory deals with individual entities with dispositional properties. I consider the properties in question as responses of measuring systems to preparation processes. The entities are only idealizations that arise from treating the measured magnitudes as properties of enduring objects. Quantum logic is a logic of measurable magnitudes, but no more.

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